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Sound generated by instability waves of supersonic flows. Part 1. Two-dimensional mixing layers

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The problem of acoustic radiation generated by a spatially growing instability wave of a supersonic two-dimensional mixing layer is studied. It is shown that at high supersonic Mach numbers the classical locally parallel-flow hydrodynamic stability theory as well as the more recent theories based on the method of multiple scales (e.g. Saric & Nayfeh 1975; Crighton & Gaster 1976; Plaschko 1979; Tam & Morris 1980) would fail to give even a first-order instability wave solution. Physically, at these high flow speeds the radiated sound field is no longer an insignificant part of the total phenomenon. The disturbances associated with the flow-instability process now extend from the mixing layer all the way to the far field. The problem is therefore global in nature. Methods of solution which are predicated on local approximations such as the classical locally parallel-flow hydrodynamic-stability theory or the method of multiple scales are hence inappropriate and inapplicable. A global solution based on the method of matched asymptotic expansions is constructed. The outer solution is valid outside the mixing layer. It provides a mathematical description of the radiated acoustic field and the pressure near field. The near field in this case consists of both the acoustic and the hydrodynamic (non-propagating) fluctuation components. The inner solution is valid inside and in the immediate vicinity of the mixing layer. Physically it represents the instability wave of the flow. Matching is carried out according to the intermediate matching principle of Van Dyke (1975) and Cole (1968). Matching terms to order unity gives the basic instability-wave solution. Matching terms to the next order gives the instability- and acoustic-wave amplitude equation. For low-Mach-number flows it is found that the present results agree with the multiple-scales solution of Tam & Morris (1980).

1. Introduction

In this and the companion paper (Part 2, Tam & Burton 1984) the phenomenon of sound generation by spatially growing instability waves in high-speed flows is investigated. This process of noise generation is most effective when the flow is supersonic relative to the ambient speed of sound. In the past, a number of investigators e.g. Sedel'nikov (1967), Tam (1971, 1972, 1975), Bishop, Ffowcs Williams & Smith (1971), Chan & Westley (1973) and Morris (1977) have suggested, on theoretical grounds, that flow instabilities could be the dominant noise-generation mechanism in supersonic jets. This idea was confirmed in a series of low-Reynolds-number supersonic-jet experiments by McLaughlin, Morrison & Troutt (1975, 1977). More recently Troutt (1978) and Troutt & McLaughlin (1982) repeated the same experiment at a moderately high Reynolds number and came to the same conclusion.

As a part of their study, Troutt & McLaughlin gently excited their jet by a glow discharge mounted flush at the nozzle exit. Under this condition hot-wire measurements in the jet flow and microphone measurements in the pressure near field outside the jet revealed the presence of a spatially growing instability wave at the forced frequency. Associated with the instability wave was an acoustic field. This acoustic field extended all the way to the boundaries of the anechoic chamber. In the present theoretical investigation, efforts will be confined primarily to the experimental situation just described. The basic reason for this is that the excited instability-wave problem is deterministic and consists of one dominant instability wave only. On the other hand, in an unexcited or natural jet the unsteady-flow components comprise a broad band of randomly mixed instability waves. To describe the gross dynamical behaviour of this band of waves a stochastic or statistical formulation becomes necessary.

Classical hydrodynamic-stability theory of a compressible flow (see e.g. Lees & Lin 1946; Lin 1953; Lees & Reshotko 1962; Mack 1965, 1975; Blumen 1970, 1971; Reshotko 1976) does not predict acoustic radiation by instability waves. In fact, the whole question had been completely ignored until the recent work of Tam & Morris (1980). The point of departure of the analysis of Tam & Morris from classical hydrodynamic stability theory is their recognition that to determine sound radiation a *global solution* of the entire wave propagation phenomenon is necessary. To understand this physically, let us examine the noise-generation processes involved. In free shear flows such as mixing layers or jets, the mean flow diverges slowly in the flow direction owing to the entrainment of ambient fluid. Over the initial region where the shear layer is thin and the mean-velocity gradient is large the amplitude of an excited instability wave grows very rapidly. As the wave propagates downstream the growth rate reduces. This is because as the flow slowly diverges the transverse velocity gradient is gradually reduced. Eventually at some point downstream the growth rate of the wave becomes zero. On propagating further downstream the wave becomes damped. Its amplitude decreases as it continues to propagate until it becomes vanishingly small. The growth and decay of the wave amplitude is extremely important to the sound-radiation process. This is especially true for instability waves with subsonic phase velocity relative to the ambient sound speed. It is well known that a subsonic wave of constant amplitude does not generate sound in a compressible medium. Such a wave has a discrete wavenumber spectrum. However, for a fixed-frequency instability wave whose amplitude undergoes growth and decay spatially its wavenumber spectrum is no longer discrete. Instead, it is broadband. Some of these broadband wave components, especially those of small wavenumbers, would actually be moving with supersonic phase velocities. These supersonic phase disturbances, by the wavy-wall analogy, will immediately lead to acoustic radiation.

To describe the growth and decay of the excited instability waves due to the slight mean-flow divergence Tam & Morris (1980) employed the method of multiple-scales asymptotic expansion (see e.g. Nayfeh 1973). In recent years the use of the method of multiple scales for the analysis of slightly non-parallel flow instability waves has become quite popular (e.g. Saric & Nayfeh 1975, 1977; Crighton & Gaster 1976; Garg & Round 1978; Plaschko 1979; Morris 1981). The procedure adopted by Tam & Morris is very similar to the method of Saric & Nayfeh (1975). However, the multiple-scales instability-wave solution, just as the classical locally parallel-flow normal-mode solution, predicts no sound radiation. As a matter of fact, all these solutions are constructed with the boundary condition that the wave disturbances decay to zero far away from the mixing layer or jet. Thus by itself the multiple-scale solution could never yield any possible acoustic field associated with the instability wave. This inadequacy of the multiple-scales asymptotic expansion solution was recognized by Tam & Morris, who showed that the asymptotic expansion is actually not uniformly valid outside the flow. Away from the flow, acoustic disturbances propagate in all directions, so that all spatial coordinates must be treated on an equal footing. Solutions obtained by the multiple-scales asymptotic expansion method, which purposely scales different spatial coordinates unevenly, are therefore inappropriate. They should not be used in the far-field region. Based on this reasoning, Tam & Morris proposed a way to construct an extended solution of the multiple-scales instability-wave solution by the method of Fourier transform. This extended solution is uniformly valid outside the mean flow. By means of this extended solution Tam & Morris (1980) were able to calculate the acoustic radiation associated with the excited instability waves in compressible two-dimensional mixing layers. Some numerical results were provided in their work and qualitative agreements with supersonic jet noise data were pointed out.

So far the method of multiple-scales asymptotic expansion has proven to be useful for the computation of instability waves at low-to-moderate-speed flows. However, the method appears to break down for high-velocity flows. Specifically, when an instability wave having supersonic phase velocity relative to the ambient speed of sound becomes neutrally stable as it propagates downstream, one is unable to use this method (and the classical hydrodynamic stability theory as well) to continue the solution into the damped region. The problem is *new* and fundamental and does not seem to have been described or encountered elsewhere before. To illustrate this 'damped supersonic wave' phenomenon let us consider the propagation of an instability wave along a supersonic jet. For convenience, we will assume that the instability wave is initiated by external disturbances of a single frequency at the nozzle exit. Just downstream of this location the shear layer is thin and the wave is unstable. For unstable waves the local instability solution (this is the lowest-order term of the multiple-scales asymptotic expansion) can readily be found by solving the appropriate local eigenvalue problem. This local eigenvalue problem is made up of the linearized compressible-flow equations and the boundedness condition. The boundedness condition is to be applied at the centreline of the jet and at a faraway location outside the jet. To simplify the analysis we will regard the fluid outside the jet as inviscid. However, it is, in fact, quite easy to show that viscosity does not play any essential role in the problem under consideration. Now outside the jet the mean flow is effectively zero so that the linearized equations of motion can be solved in closed form. With respect to a cylindrical coordinate system (r, θ, x) centred at the nozzle exit and the x-axis pointing in the direction of flow, the lowest-order pressure perturbation associated with the instability wave of the nth azimuthal mode and angular frequency ω can be written in terms of the nth-order Hankel function of the first kind $H_n^{(1)}$ as

$$p(r,\theta,x,t) = \mathrm{Re}\left\{AH_n^{(1)}\left(\mathrm{i}r\left(\alpha^2-\omega^2/a_\infty^2\right)^{\frac{1}{2}}\right)\exp\mathrm{i}\left(\int_0^x \alpha\,\mathrm{d}x+n\theta-\omega t\right)\right\}, \qquad (1.1)$$

where Re{ } = the real part of { }.

In (1.1) A is the slowly varying wave amplitude, α is the slowly varying eigenvalue (wavenumber) of the instability wave and a_∞ is the ambient speed of sound. To ensure that the eigenfunction is bounded or represents an outgoing wave as $r \rightarrow \infty$, the branch cuts of the square-root function in the argument of the Hankel function will be chosen to satisfy the condition

$$-\tfrac{1}{2}\pi \leqslant \mathrm{Arg}\left(\alpha^2-\omega^2/a_\infty^2\right)^{\frac{1}{2}} < \tfrac{1}{2}\pi. \qquad (1.2)$$

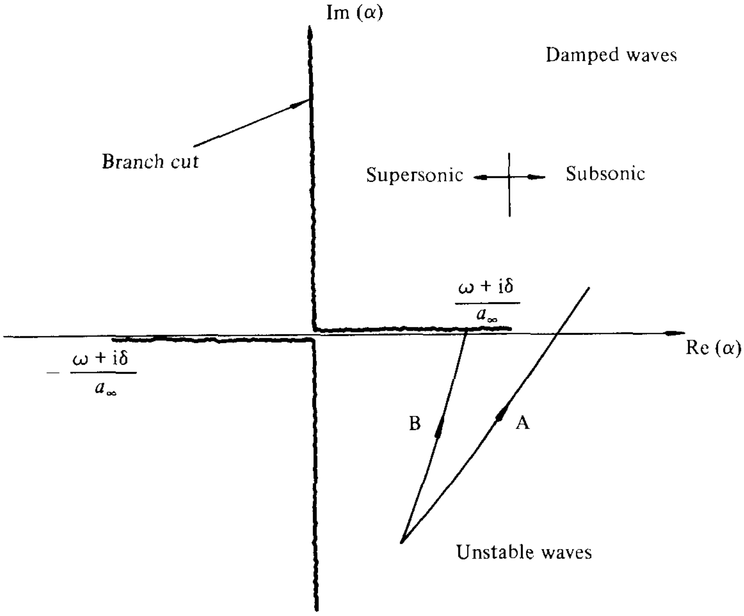

FIGURE 1. The complex α-plane showing branch cuts (limit $\delta \to 0^+$) and possible trajectories of the local eigenvalues of an instability wave with supersonic phase velocity initially.

These branch cuts extend from the branch points $\alpha = \omega/a_\infty$ and $-\omega/a_\infty$ to infinity in the complex α-plane as shown in figure 1. It is straightforward to see that an eigenvalue at the branch point $\alpha = \omega/a_\infty$ corresponds to a wave travelling with exactly the ambient speed of sound in the jet flow direction. Immediately downstream of the nozzle exit the excited wave is unstable. That is, the local eigenvalue α has a negative imaginary part. For a very-high-speed jet the local phase velocity of the instability wave in this region is supersonic with respect to the ambient speed of sound so that the real part of α is positive but less than ω/a_∞. In other words the local eigenvalue lies initially at a point in the fourth quadrant of the α-plane as depicted in figure 1. As the instability wave propagates downstream its growth rate reduces as discussed before. This means that at this new location of the jet the imaginary part of the local eigenvalue is less negative. It is, therefore, represented by a point closer to the real α-axis. On following the propagation of the instability wave downstream one finds that the local eigenvalue traces a trajectory in the α-plane which moves closer and closer to the real axis. Since the mean velocity of the jet gradually decreases in the flow direction the phase velocity of the instability wave would eventually become subsonic. This may occur before the wave reaches the region of the jet where it becomes damped. When this happens the trajectory of the local eigenvalue would cross the real α-axis at a point lying to the right of the branch cut as indicated by path 'A' of figure 1. In this case the use of the method of multiple scales to obtain a global wave solution encounters no difficulty as was in the work of Tam & Morris (1980). However, for very-high-speed flows the phase velocity of the excited instability wave could remain supersonic even when the wave enters the part of the jet where it is damped. A wave of this kind would have a trajectory similar to that of path 'B' in figure 1. The trajectory terminates at the branch cut. Since one is not allowed to cross the branch cut it is therefore impossible to continue the eigensolution downstream of this point (and still satisfy the boundedness condition

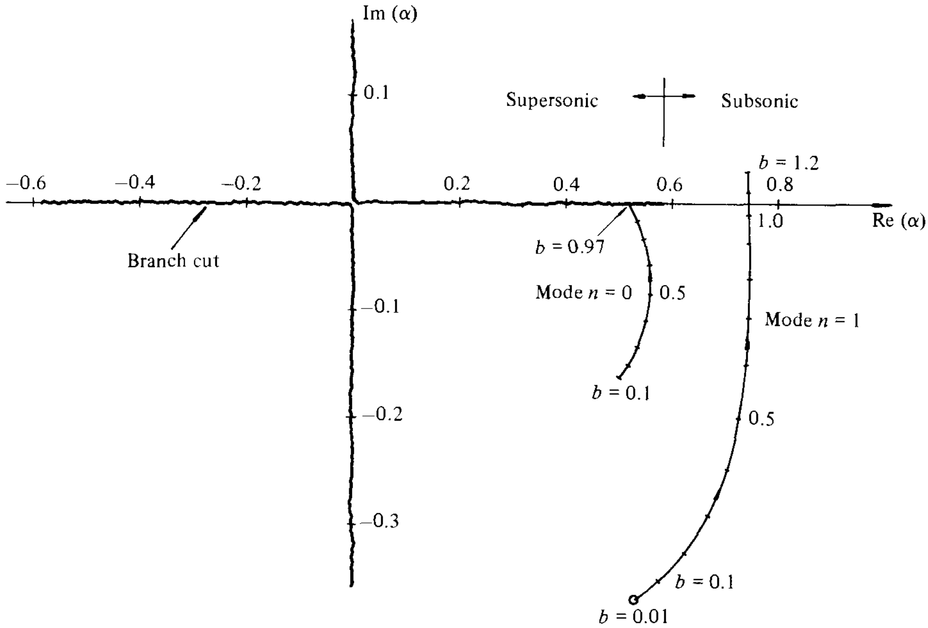

FIGURE 2. The complex α-plane showing the trajectories of the local eigenvalues of the axisymmetric ($n = 0$) and the helical ($n = 1$) instability wave modes of a cold supersonic jet. Mach number 1.5, Strouhal number = 0.15, b = half-width of the mixing layer of the jet. All quantities are non-dimensionalized by the radius of the jet at the nozzle exit.

at $r \rightarrow \infty$). In addition, it is also important to point out that right at the branch cut the wave is neutrally stable. At this neutral stable point the local eigenfunction has the form of an undamped cylindrical wave at $r \rightarrow \infty$. In the method of multiple-scales asymptotic expansion (see Nayfeh 1973; Saric & Nayfeh 1975; Crighton & Gastor 1976; Tam & Morris 1980; Morris 1981) this eigenfunction is used to compute the integrals of the solvability condition at the next stage of the analysis. The solvability condition is crucial to the success of the method, as it determines the slowly varying wave amplitude. But at the supersonic neutral stable condition the integrals, now having the neutrally stable eigenfunction in their integrands, become divergent. Thus when the branch cut is reached it is impossible to calculate the slowly varying wave amplitude as well. Clearly therefore one is forced to conclude that the method of multiple-scales asymptotic expansion would break down completely when a supersonic instability wave becomes damped.

We will now demonstrate by a concrete numerical example that this 'damped supersonic wave' phenomenon can occur even at moderately high jet Mach number (otherwise it would be of no practical concern to us). Figure 2 shows the trajectories of the eigenvalues of the axisymmetric mode ($n = 0$) and the helical ($n = 1$) mode instability waves of a cold supersonic jet of Mach number 1.5 excited at a Strouhal number 0.15. In calculating these trajectories the mean-velocity profile of the jet has been assumed to consist of a uniform flow in the core region and a half-Gaussian velocity profile in the mixing layer. The core radius and the half-width b of the mixing layer are further assumed to be related by the requirement of conservation of total momentum flux, a condition verified experimentally (see Eggers 1966). This velocity profile has been found to approximate the measured data quite well in many instances (e.g. Troutt 1978: also we have tested this profile against the data of Eggers 1966; Lau, Morris & Fisher 1979; Lau 1981). As can be seen in this figure, the helical instability wave has subsonic phase velocity downstream of the location where the half-width of the mixing layer is slightly greater than 10 % of the nozzle exit radius. The trajectory of this wave crosses the real axis at a subsonic point so that the method of multiple scales is useful for calculating a global solution. On the other hand, the phase velocity of the axisymmetric instability wave is supersonic for all values of b (non-dimensionalized by the nozzle exit radius) up to 0.97, at which the wave becomes neutrally stable. Beyond this value of b no eigensolution of the same family satisfying the boundedness condition at $r \rightarrow \infty$ can be found. That is to say, the method of multiple scales would not be able to provide an adequate mathematical description of the spatial evolution of the axisymmetric instability wave of a round jet even at a modest supersonic Mach number of 1.5.

There are two primary objectives in the present investigation. The first is to develop a new mathematical procedure capable of calculating the global solution of excited instability waves in high-speed flows. Here global solution refers not only to the instability-wave solution in the flow but also its associated acoustic field as well. The second objective is to verify, as unambigously as possible, the suggestion that large coherent disturbances are important sources of sound in supersonic flows by comparing the calculated flow and acoustic field of the instability wave solution with the experimental measurements of Troutt (1978) and Troutt & McLaughlin (1982). Troutt's experiment (see also Troutt & McLaughlin 1982) on a 2.1 Mach-number axisymmetric cold-air jet is selected for comparison because it provides the only available set of reliable laboratory flow and acoustic data at a Reynolds number which can be considered closely approximating that of a practical jet. It is worth while to point out that most previous experiments were carried out with the purpose of measuring either the instability-wave characteristics in the flow or the spectra and directivity of the acoustic field of a jet but never both at the same time. With respect to our first objective, we note that the global solution of Tam & Morris (1980) including the acoustic field for two-dimensional mixing layers is correct as far as subsonic phase velocity instability waves are concerned. So it would provide an important check to any new method of solution. Further, as can be seen in their paper, many separate steps are necessary to construct the overall solution. It would be best to avoid complicating the presentation of our new method of solution by very involved algebraic expressions. Because of these many considerations we feel that it is prudent to report the results of our investigation in two parts. Part 1 outlines the new method of solution for the problem of sound radiation by instability waves in two-dimensional mixing layers. Unlike the supersonic jet problem where complicated special functions are involved the algebra in this case is rather straightforward. Moreover, working out the solution of this problem allows a direct comparison with the results of Tam & Morris (1980). In Part 2 numerical results of the excited instability waves of a cold supersonic jet will be presented. These theoretical predictions are then compared with the hot-wire measurements of Troutt (1978) and Troutt & McLaughlin (1982) inside the jet flow and their microphone measurements in the acoustic field.

The main reason that the method of multiple scales fails to provide a damped supersonic wave solution is the non-existence of a local solution of the same family which remains bounded at large distance from the sheared region of the mean flow. However, Tam & Morris (1980) have already shown that the multiple-scales solution is not actually valid in these faraway regions. Therefore a correct way to treat the problem might simply consist of the relaxation of the boundedness condition at infinity, where the solution is not valid in the first place. In fluid mechanics, problems

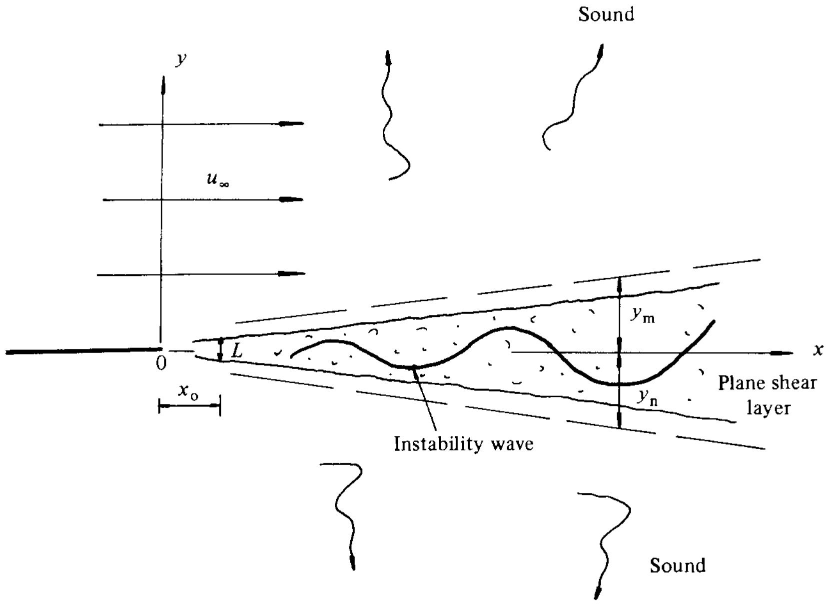

FIGURE 3. Instability wave in a two-dimensional mixing layer.

involving non-uniformity and/or non-boundedness of asymptotic expansions of the kind encountered here are quite common. These problems are generally referred to as being 'singular'. In the literature several methods of treating singular perturbation problems are known. Here we will show that the method of matched asymptotic expansions when suitably adopted can provide valid global solutions of excited instability waves at high-speed flows. Within the framework of the method of matched asymptotic expansions a separate solution, the outer expansion, which is valid outside the shear layer, will be constructed. It will become clear later that this outer expansion is the same as the extended solution of Tam & Morris when the phase velocity of the instability wave is subsonic. Now instead of requiring boundedness of solution at infinity, the outer boundary condition of the instability wave solution (this is the first term of the inner expansion according to the usual terminology of the method of matched asymptotic expansion) is that it matches the outer expansion in the overlap domain of validity. This overlap domain exists just outside the sheared region of the mean flow. Although a number of non-trivial steps are necessary to accomplish this yet, in a nutshell, this is the way we propose to resolve the branch cut problem of the 'damped supersonic wave' phenomenon.

To apply the method of matched asymptotic expansions, the first important step is the choice of the appropriate inner and outer variables. For slowly divergent free shear flows such as two-dimensional mixing layers the rate of spread ϵ is usually a small parameter. If x is the coordinate in the direction of flow and y is the coordinate in the direction of the mean shear gradient (see figure 3) then the mean flow is a function of y and the slow variable s, where $s = \epsilon x$. The set of inner variables suitable for the description of the excited instability waves in the mixing layer is also (s, y). It turns out that to the lowest order in ϵ the differential equation does not involve derivatives in s, so that s becomes a local parameter. Therefore this selection of inner variables effectively ensures that the lowest-order solution is identical with that of the classical locally parallel-flow approximation. Before choosing the outer variables it would be helpful to recall that the overall spatial growth and decay of the wave amplitude is crucial to sound radiation. Clearly this gradual amplitude variation in the flow direction is a function of the slow variable s. This physical consideration suggests that the appropriate outer variable in the flow direction is s. Further, since sound propagates without inherently preferred direction in the far field, the spatial variables in this region must be scaled in the same manner in all directions. Hence a suitable set of outer variables for the present problem appears to be $(s, \bar{y})$, where $\bar{y} = \epsilon y$.

In §§3 and 4 the inner and outer asymptotic expansions corresponding to an excited instability wave in a two-dimensional mixing layer and its associated acoustic fields are constructed in terms of the inner and outer spatial variables (s, y) and $(s, \bar{y})$. Matching of these solutions is carried out in §§5 and 6. It is to be noted that the overlap domains lying just outside the shear layer where both the inner and outer asymptotic expansions are valid turns out to be quite narrow. Because of this, matching is performed in terms of intermediate variables according to the intermediate matching principle (see Van Dyke 1975, §5.8). Resolution of the problem of 'damped supersonic wave' by means of analytic continuation is discussed in §7. Finally, the present inner and outer asymptotic expansions of excited instability waves with subsonic phase velocity are shown to be identical with the results of Tam & Morris (1980).

2. The physical problem

The spatial evolution of a small-amplitude instability wave in a pre-existing two-dimensional supersonic mixing layer as shown in figure 3 is considered. The instability wave is assumed to be initiated by a localized external excitation of frequency Ω near the trailing edge of the splitter plate. On account of the boundary layer on the splitter plate the mixing layer has an initial thickness of L at a distance x_0 downstream of the virtual origin. The static pressure will be taken to be constant throughout the flow. Since the mean flow is dynamically unstable even in the absence of viscosity, the instability wave and its acoustic field will be assumed to satisfy the linearized inviscid, compressible equations of motion. These equations are the linearized continuity, momentum and energy equations together with the equation of state. In dimensionless form using the freestream quantities U_∞, ρ_∞ as the velocity and density scales, L (the initial thickness of the mixing layer) as lengthscale and L/U_∞, $\rho_\infty U_\infty^2$ as the time and pressure scales respectively, these equations can readily be written as

$$\frac{\partial u_i'}{\partial t} + \bar{u}_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x_i}, \tag{2.1}$$

$$\frac{\partial p'}{\partial t} + \bar{u}_j \frac{\partial p'}{\partial x_j} + \frac{1}{M^2} \frac{\partial u_j'}{\partial x_j} + \gamma p' \frac{\partial \bar{u}_j}{\partial x_j} = 0, \tag{2.2}$$

where primes denote fluctuating quantities, γ is the ratio of specific heats and M is the freestream Mach number. The mean flow of the two-dimensional mixing layer is a slowly varying function of the coordinate in the flow direction. Measured mean-flow profiles will be used. Experimental data obtained by Liepmann & Laufer (1947) and at supersonic Mach numbers by Hill & Page (1969) show that the mean velocities may be presented analytically in the form

$$\bar{\boldsymbol{u}} = (\bar{u}(y/s), \epsilon\bar{v}(y/s), 0), \tag{2.3a}$$

where

$$\bar{u} = 1, \quad \bar{v} = 0 \quad (y \geqslant y_{\mathrm{m}}), \tag{2.3b}$$

$$\bar{u} = 0, \quad \epsilon\bar{v} = \epsilon\bar{v}_{\infty} \quad (y \leqslant -y_{\mathrm{n}}). \tag{2.3c}$$

This form of the mean-flow profile was also used by Tam & Morris (1980). In (2.3*a–c*) $s = \epsilon x$, where ϵ is a measure of the rate of spread of the mixing layer. Numerically ϵ, which is a function of Mach number, is less than 0.1 and will be regarded here as a small parameter of the problem. The two velocity components of (2.3*a*) are related by the continuity equation. For freestream Mach number up to about two, a physically realistic approximation is to assume that the static temperature, and hence the density also, is constant in the mean continuity equation. This simplifying assumption adopted by Tam & Morris can, however, be easily relaxed. To account for the compressibility effects in the linearized momentum and energy equations (2.1) and (2.2), the mean density $\bar{\rho}$ will be taken to be related to the mean velocity by Crocco's relation.

Since the instability wave is initiated by external excitation of frequency ω, the solution of (2.1) and (2.2) will have a time-dependent factor of the form $\exp(-\mathrm{i}\omega t)$. Thus on separating out this time factor the governing equations for the spatial part of the solution are

$$-\mathrm{i}\omega u_i + \bar{u}_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x_i}, \tag{2.4}$$

$$-\mathrm{i}\omega p + \bar{u}_j \frac{\partial p}{\partial x_j} + \frac{1}{M^2}\frac{\partial u_j}{\partial x_j} + \gamma p \frac{\partial \bar{u}_j}{\partial x_j} = 0. \tag{2.5}$$

These are the equations to be solved in the mixing layer as well as in the acoustic field. Because of the variable coefficients involved a simple analytical solution which is valid everywhere cannot be found. In the rest of this paper a solution that represents physically a spatially growing instability wave and its associated acoustic field will be constructed by the method of matched asymptotic expansions. The inner solution is to be valid inside the mixing layer and in the adjacent near pressure field outside the mixing layer. The fluctuation in the near field consist of both hydrodynamic (non-propagating) and acoustic components. The outer solution is to be valid in the acoustic far field and the near field. These two solutions are to be matched in the near pressure field where both solutions are valid.

3. The inner solution

As discussed in §1, the appropriate inner variables are y and the slow variable s. Physically the inner solution models a wave that propagates through a slightly inhomogeneous medium formed by the mean flow. Problems of this kind have been studied extensively before (e.g. Whitham 1974, §11.8). Such a wave may be represented analytically in the form of an asymptotic expansion with ϵ, the rate of spread of the mixing layer, as the small parameter.

$$\begin{bmatrix} u \\ v \\ p \end{bmatrix} = \sum_{n=0}^{\infty} \epsilon^n \begin{bmatrix} u_n(s,y) \\ v_n(s,y) \\ p_n(s,y) \end{bmatrix} \mathrm{e}^{\mathrm{i}\theta(s)/\epsilon}. \tag{3.1}$$

In (3.1) $s = \epsilon x$ is the slow variable and $\theta(s)$ is the phase function. For convenience we will denote the derivative of $\theta(s)$ by $\alpha(s)$; that is,

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \alpha(s). \tag{3.2}$$

Physically $\alpha(s)$ is the local (complex) wavenumber.

Substitution of (3.1) into (2.4) and (2.5) and partitioning terms according to powers of ϵ and then eliminating all other variables in favour of pressure p gives, to order unity,

$$\frac{\partial^2 p_0}{\partial y^2} + \left[\frac{2\alpha}{\omega-\alpha\bar{u}}\frac{\partial \bar{u}}{\partial y} - \frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial y}\right]\frac{\partial p_0}{\partial y} - [\alpha^2 - \bar{\rho}M^2(\omega-\alpha\bar{u})^2]\,p_0 = 0. \tag{3.3}$$

In general the order ϵ^n equation is

$$\frac{\partial^2 p_n}{\partial y^2} + \left[\frac{2\alpha}{\omega-\alpha\bar{u}}\frac{\partial \bar{u}}{\partial y} - \frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial y}\right]\frac{\partial p_n}{\partial y} - [\alpha^2 - \bar{\rho}M^2(\omega-\alpha\bar{u})^2]\,p_n = \chi_n \quad (n = 1, 2, 3, \ldots). \tag{3.4}$$

The inhomogeneous term χ_n on the right-hand side of (3.4) contains only lower-order quantities, i.e. $p_0, p_1, \ldots, p_{n-1}$.

Now for $y > y_{\mathrm{m}}$, on account of the mean flow given by (2.3*b*), (3.3) reduces to

$$\frac{\partial^2 p_0}{\partial y^2} + [M^2(\omega-\alpha)^2 - \alpha^2]\,p_0 = 0. \tag{3.5}$$

Two linearly independent solutions of (3.5) are

$$p_0 = \mathrm{e}^{\mathrm{i}\lambda_+ y}, \qquad p_0 = \mathrm{e}^{-\mathrm{i}\lambda_+ y}, \tag{3.6}$$

where

$$\lambda_+(\alpha) = [M^2(\omega-\alpha)^2 - \alpha^2]^{\frac{1}{2}}. \tag{3.7}$$

At this time any branch of the square root of the right-hand side of (3.7) may be used. However, to facilitate the process of matching of solutions which will be carried out later, it is advantageous to choose the branch cuts for λ_+ in the α-plane as shown in figure 4. This choice of the branch cut assures that $0 \leqslant \arg(\lambda_+) < \pi$ in the entire complex α-plane.

Let $\zeta_1(s, y)$ and $\zeta_2(s, y)$ be two linearly independent solutions of (3.3) such that for $y > y_{\mathrm{m}}$

$$\zeta_1 = \mathrm{e}^{\mathrm{i}\lambda_+ y}, \qquad \zeta_2 = \mathrm{e}^{-\mathrm{i}\lambda_+ y}. \tag{3.8}$$

The general solution of the zeroth-order inner solution may be written as

$$p_0(y, s) = A_0(s)\,\zeta_1(s, y) + B_0(s)\,\zeta_2(s, y). \tag{3.9}$$

In (3.9) $A_0(s)$ and $B_0(s)$ are unknown amplitude functions.

In the region just below the mixing layer, i.e. $y < -y_{\mathrm{n}}$, (3.3) becomes

$$\frac{\partial^2 p_0}{\partial y^2} - (\alpha^2 - \rho_\infty \omega^2 M^2)\,p_0 = 0. \tag{3.10}$$

Here ρ_∞ is the dimensionless mean fluid density in the region $y < -y_{\mathrm{n}}$. Two linearly independent solutions of (3.10) are $\mathrm{e}^{\lambda_- y}$ and $\mathrm{e}^{-\lambda_- y}$, where

$$\lambda_- = (\alpha^2 - \rho_\infty \omega^2 M^2)^{\frac{1}{2}}. \tag{3.11}$$

Again the choice of the branch cuts of the square-root function in λ_- is quite arbitrary at this time. Ultimately it is decided by the process of matching the inner and the outer solutions. In anticipation of this we will choose the branch cut as shown in figure

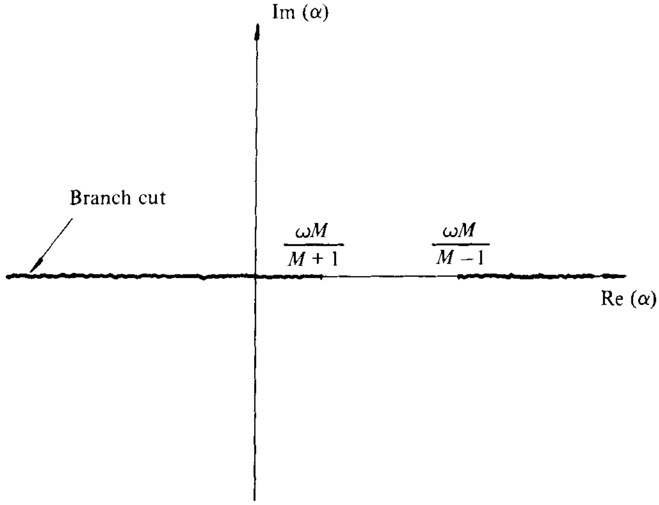

FIGURE 4. Branch cuts for the function $\lambda_+(\alpha)$.

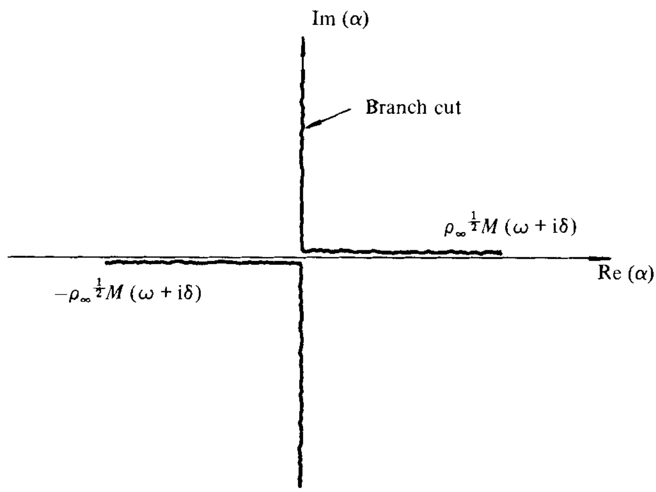

FIGURE 5. Branch cuts for the function $\lambda_-(\alpha)$ (limit $\delta \to 0^+$).

5. To avoid confusion the frequency parameter ω should be treated as a Fourier–Laplace transform variable with an infinitesimally small positive imaginary part. Formally we will take the branch points of λ_- to be at

$$\alpha = \lim_{\delta \to 0^+} \pm \rho_\infty^{\frac{1}{2}} M(\omega + \mathrm{i}\delta).$$

This choice of the branch cuts ensures that $-\frac{1}{2}\pi \leqslant \arg(\lambda_-) < \frac{1}{2}\pi$ in the complex α-plane.

Now for $y < -y_n$ the solutions $\zeta_1(s, y)$ and $\zeta_2(s, y)$ will, in general, each be a linear combination of the two linearly independent solutions of (3.10). This may be expressed explicitly as

$$\left.\begin{aligned} \zeta_1(s,y) &= \beta_1(\alpha)\,\mathrm{e}^{\lambda_- y} + \bar{\beta}_1(\alpha)\,\mathrm{e}^{-\lambda_- y}, \\ \zeta_2(s,y) &= \beta_2(\alpha)\,\mathrm{e}^{\lambda_- y} + \bar{\beta}_2(\alpha)\,\mathrm{e}^{-\lambda_- y}, \end{aligned}\right\} \quad (y < -y_n). \tag{3.12}$$

In (3.12) the constants β_1, $\bar{\beta}_1$, β_2 and $\bar{\beta}_2$ are functions of the wavenumber α.

To sum up, from (3.1) and (3.9) the one-term inner expansion denoted by a subscript i is

$$p_{\mathrm{i}} = [A_0(s)\,\zeta_1(s,y) + B_0(s)\,\zeta_2(s,y) + O(\epsilon)]\,\mathrm{e}^{\mathrm{i}\theta(s)/\epsilon}. \tag{3.13}$$

The behaviour of the functions ζ_1 and ζ_2 for $y > y_{\mathrm{m}}$ and $y < -y_{\mathrm{n}}$ are given by equations (3.8) and (3.12) respectively. In this solution the functions $A_0(s)$, $B_0(s)$ and $\alpha(s) = \mathrm{d}\theta/\mathrm{d}s$ are still unknown. They will be determined later. For damped waves the functions $\zeta_1(s,y)$ and $\zeta_2(s,y)$ are to be constructed by the deformed contour integration method to be carried out in the complex y-plane as discussed by Tam & Morris (1980).

4. The outer solution

There are two outer solutions in this problem. One is to be valid in the region $y > y_{\mathrm{m}}$ and the other in the region $y < -y_{\mathrm{n}}$. To distinguish these two different solutions we will refer to the former as the upper outer solution and the latter as the lower outer solution. In §1 the choice of suitable outer variables was discussed. It was argued that in regions far away from the mixing layer the acoustic wave should propagate with no preferential direction. To comply with this physical requirement the spatial coordinates must, therefore, have the same scaling in all directions. The appropriate outer variables were found to be $s = \epsilon x$ and $\bar{y} = \epsilon y$. In the region where the upper outer solution is valid the governing equations (2.4) and (2.5) written in the outer variables become

$$\begin{aligned} -\mathrm{i}\frac{\omega}{\epsilon}u + \frac{\partial u}{\partial s} &= -\frac{\partial p}{\partial s},\\ -\mathrm{i}\frac{\omega}{\epsilon}v + \frac{\partial v}{\partial s} &= -\frac{\partial p}{\partial \bar{y}},\\ -\mathrm{i}\frac{\omega}{\epsilon}M^2p + M^2\frac{\partial p}{\partial s} + \frac{\partial u}{\partial s} + \frac{\partial v}{\partial \bar{y}} &= 0. \end{aligned} \tag{4.1}$$

In the limit $\epsilon \rightarrow 0$ with the outer variables fixed (4.1) is exact to all powers of ϵ. By eliminating u and v from (4.1) a single equation for p may be found:

$$M^2\left(-\mathrm{i}\frac{\omega}{\epsilon} + \frac{\partial}{\partial s}\right)^2 p - \left(\frac{\partial^2 p}{\partial s^2} + \frac{\partial^2 p}{\partial \bar{y}^2}\right) = 0. \tag{4.2}$$

Since the domain of this solution extends to $\bar{y} \rightarrow \infty$, it must satisfy the boundedness or outgoing-wave condition there. In addition, this upper outer solution must match the inner solution according to the matching principle.

A formal exact solution of (4.2) satisfying the boundedness or outgoing wave condition at $\bar{y} \rightarrow \infty$ can easily be found by the method of Fourier transform. To construct this solution, the first step is to apply a Fourier transform to the s-variable in (4.2). The transformed equation is an ordinary differential equation which depends on the variable $\bar{y}$ alone. This equation can be solved in a straightforward manner. By inverting the Fourier transform, the upper outer solution denoted by a subscript o and a superscript u may be written as

$$p_{\mathrm{o}}^{\mathrm{u}}(s,y) = \int_{-\infty}^{\infty} g(k,\epsilon)\exp\left\{\mathrm{i}[M^2(\omega-\epsilon k)^2 - k^2\epsilon^2]^{\frac{1}{2}}\frac{\bar{y}}{\epsilon} + \mathrm{i}ks\right\}\mathrm{d}k. \tag{4.3}$$

In (4.3) k is the Fourier transform variable. The function $g(k,\epsilon)$, without loss of generality, will be considered as a Fourier transform of an arbitrary function $\tilde{A}(s,\epsilon)$ as follows:

$$g(k,\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{A}(s,\epsilon)\,\mathrm{e}^{\mathrm{i}\theta(s)/\epsilon-\mathrm{i}ks}\,\mathrm{d}s. \tag{4.4}$$

This representation of $g(k,\epsilon)$, suggested by the work of Tam & Morris (1980), will facilitate the process of matching solutions later on. In order that the boundedness or outgoing wave condition at $\bar{y}\to\infty$ be satisfied, the branch cuts of the square-root function $[M^2(\omega-\epsilon k)^2-k^2\epsilon^2]^{\frac{1}{2}}$ in (4.4) must be specified properly. This function depends on the variable $\eta=\epsilon k$. Thus in the complex η-plane the branch cuts are to be inserted such that $0 \leqslant \arg[M^2(\omega-\eta)^2-\eta^2]^{\frac{1}{2}} < \pi$. This choice of the branch cuts assures that as $\bar{y}\to\infty$ the upper outer solution of (4.3) either goes to zero or represents an outgoing wave for all values of η in the complex plane. As a function of η the square-root function can easily be recognized to be identical with $\lambda_+(\eta)$ (see (3.7)). The branch cuts of $\lambda_+(\alpha)$ and $\lambda_+(\eta)$ are the same in the α- and η-planes (see figure 4).

In the region $y<-y_{\mathrm{n}}$ a formal exact lower outer solution may be constructed following the same steps as above. This solution which will be denoted by a subscript o and a superscript ℓ, has the form

$$p_{\mathrm{o}}^{\ell}(s,\bar{y}) = \int_{-\infty}^{\infty} \hat{g}(k,\epsilon)\exp\left\{\mu(\epsilon k,\epsilon)\frac{\bar{y}}{\epsilon}+\mathrm{i}ks\right\}\mathrm{d}s, \tag{4.5}$$

where the function $\hat{g}(k,\epsilon)$ is given by the Fourier transform of an arbitrary function $\hat{A}(s,\epsilon)$:

$$\hat{g}(k,\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{A}(s,\epsilon)\,\mathrm{e}^{\mathrm{i}\theta(s)/\epsilon-\mathrm{i}ks}\,\mathrm{d}s. \tag{4.6}$$

The function $\mu(\epsilon k,\epsilon)$ in (4.5) when written out explicitly is

$$\mu = \frac{\mathrm{i}\omega(\epsilon\bar{v}_{\infty})\rho_{\infty}M^2}{1-(\epsilon\bar{v}_{\infty})^2\rho_{\infty}M^2} + \frac{[(\epsilon k)^2-\rho_{\infty}M^2\omega^2/(1-(\epsilon\bar{v}_{\infty})^2\rho_{\infty}M^2)]^{\frac{1}{2}}}{[1-(\epsilon\bar{v}_{\infty})^2\rho_{\infty}M^2]^{\frac{1}{2}}}. \tag{4.7}$$

The branch of the square-root function in the ξ-plane ($\xi=\epsilon k$) to be used in (4.7) is

$$-\tfrac{1}{2}\pi \leqslant \arg\left[\xi^2-\frac{\rho_{\infty}M^2\omega^2}{(1-(\epsilon\bar{v}_{\infty})^2\rho_{\infty}M^2)}\right]^{\frac{1}{2}} < \tfrac{1}{2}\pi. \tag{4.8}$$

This choice ensures that the boundedness or outgoing-wave condition at $\bar{y}\to-\infty$ is satisfied.

To sum up, the upper and lower outer solutions are given by (4.3)–(4.6). In these solutions the functions $\tilde{A}(s,\epsilon)$ and $\hat{A}(s,\epsilon)$ are arbitrary. These functions are to be determined by the process of matching, which will be discussed in §5.

5. Matching of solutions

In many fluid-mechanics problems that are solvable by the method of matched asymptotic expansions (see Van Dyke 1975, §5.8 and also note 5) the overlap domain of validity may become vanishingly small in the outer or the inner limit. When this occurs the usual inner- and outer-limit matching process would fail. Fortunately, according to Kaplun's (1967) concept of continuum of intermediate limit the problem can generally be resolved by the use of intermediate matching, a procedure always favoured by Cole (1968). It turns out that the problem under consideration falls into this category. Hence to match the solutions we will follow the intermediate matching principle of Van Dyke and Cole.

Let us introduce an intermediate variable $\tilde{y}$ defined by

$$\tilde{y} = \epsilon^{1/N} y, \tag{5.1}$$

where N is a large positive number ($N \gg 1$). In addition the slow coordinate variable s will again be used. In the present problem the outer solutions given by (4.3)–(4.6) are exact to all powers of ϵ. Therefore the intermediate solution can easily be obtained by changing the coordinates of the outer solutions to the intermediate variables, i.e. replacing $\bar{y}$ by $\epsilon^{1-1/N}\tilde{y}$ in (4.3)–(4.6). Since the outer and intermediate solutions are identical, to match the solutions (according to the intermediate matching principle), it is only necessary to show that the asymptotic expansion of the intermediate solution agrees with the intermediate limit of the inner solution to the appropriate order.

To develop the intermediate asymptotic expansion we will first rewrite (4.3) and (4.4), the upper intermediate solution, in the following form obtained by changing the integration variable to η, where $\eta = \epsilon k$:

$$p_0^{\mathrm{u}}(s, \tilde{y}) = \int_{-\infty}^{\infty} \frac{1}{\epsilon} g\left(\frac{\eta}{\epsilon}, \epsilon\right) \exp\left[\mathrm{i}\lambda_+(\eta)\, \tilde{y}\epsilon^{-1/N} + \frac{\mathrm{i}\eta s}{\epsilon}\right] \mathrm{d}\eta, \tag{5.2}$$

$$g\left(\frac{\eta}{\epsilon}, \epsilon\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(s, \epsilon) \exp\left[\frac{\mathrm{i}(\theta(s) - \eta s)}{\epsilon}\right] \mathrm{d}s. \tag{5.3}$$

It will be assumed that the unknown function $\tilde{A}(s,\epsilon)$ has an asymptotic expansion of the form

$$\tilde{A}(s, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \tilde{A}_n(s). \tag{5.4}$$

On substituting (5.4) into (5.3), the integral, in the limit $\epsilon \to 0$ with η fixed, may be evaluated asymptotically by the well-known saddle-point method. The phase function $F(s)$ of the integral is

$$F(s) = \mathrm{i}(\theta(s) - \eta s). \tag{5.5}$$

The saddle points are given by the roots of $F'(s) = 0$ or, from (3.2),

$$\alpha(s) - \eta = 0. \tag{5.6}$$

We will assume that the two-dimensional mixing layer has only one unstable wave mode, which implies, as will become clear later, that (5.6) has only one root. Let us denote this root by

$$s = \bar{s}(\eta) \quad \text{where} \quad \alpha(\bar{s}) = \eta. \tag{5.7}$$

Dingle (1973) has worked out the formulas for the full asymptotic expansion of an integral of the form that appeared in (5.3). On following his formulas, the first few terms of the asymptotic expansion can easily be found. They are

$$g\left(\frac{\eta}{\epsilon}, \epsilon\right) \sim \left[\frac{\epsilon}{-\mathrm{i}2\pi\alpha'(\bar{s})}\right]^{\frac{1}{2}} \exp\left[\mathrm{i}\frac{(\theta(\bar{s}) - \eta\bar{s})}{\epsilon}\right] D(\eta, \epsilon), \tag{5.8a}$$

where

$$D(\eta, \epsilon) = \tilde{A}_0(\bar{s}) + \epsilon\left\{\tilde{A}_1(\bar{s}) + \frac{\mathrm{i}}{24(\alpha')^3}[\tilde{A}_0(5(\alpha'')^2 - 3\alpha'\alpha''') - 12\tilde{A}_0'\alpha'\alpha'' + 12\tilde{A}_0''(\alpha')^2]\right\} + O(\epsilon^2), \tag{5.8b}$$

$$\alpha' = \frac{\mathrm{d}\alpha(\bar{s})}{\mathrm{d}\bar{s}}, \quad \alpha'' = \frac{\mathrm{d}^2\alpha(\bar{s})}{\mathrm{d}\bar{s}^2}, \quad \tilde{A}_0' = \frac{\mathrm{d}\tilde{A}_0(\bar{s})}{\mathrm{d}\bar{s}}, \quad \tilde{A}_0'' = \frac{\mathrm{d}^2\tilde{A}(\bar{s})}{\mathrm{d}\bar{s}^2}.$$

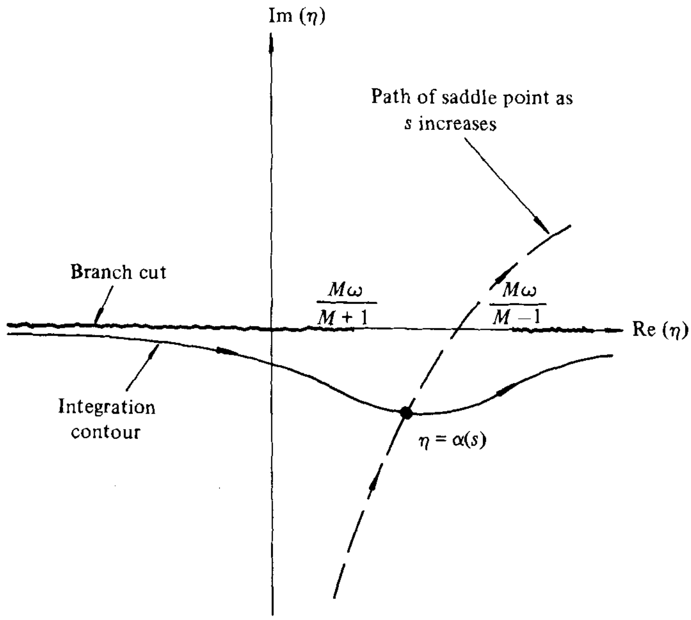

FIGURE 6. The η-plane showing the branch cuts of $\lambda_+(\eta)$, the saddle point $\eta = \alpha(s)$ and its trajectory as s increases, and the deformed integration contour.

On substituting (5.8) into (5.2), the infinite integral can again be evaluated asymptotically by the saddle-point method. Here we have

$$p_0^u(s, \tilde{y}) \underset{\substack{s,\,\tilde{y}\text{ fixed}\\ \epsilon \to 0}}{\sim} \frac{1}{(2\pi\epsilon)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\exp\left[\mathrm{i}\dfrac{\theta(\bar{s}) - \eta(\bar{s}-s)}{\epsilon} + \mathrm{i}\lambda_+(\eta)\,\tilde{y}\epsilon^{-1/N}\right]}{(-\mathrm{i}\alpha')^{\frac{1}{2}}} D(\eta, \epsilon)\,\mathrm{d}\eta. \quad (5.9)$$

The saddle point in this case is given by the root of

$$\frac{\mathrm{d}}{\mathrm{d}\eta}[\theta(\bar{s}) - \eta(\bar{s}-s)] = 0$$

or

$$[\alpha(\bar{s}) - \eta]\frac{\mathrm{d}\bar{s}}{\mathrm{d}\eta} - (\bar{s}-s) = 0. \quad (5.10)$$

But, from the definition of $\bar{s}$ given by (5.7), the first term of (5.10) is zero, so that

$$\bar{s}(\eta) = s. \quad (5.11)$$

Equation (5.11) gives the implicit relation of η as a function of s. However, when this condition holds, the explicit relation is given by (5.7); that is,

$$\eta = \alpha(s). \quad (5.12)$$

(Note that the s-variable in (5.12) is the slow coordinate variable in the flow direction, whereas the s in (5.7) is the integration variable of the saddle-point method.) Notice that the location of the saddle point $\eta = \alpha(s)$ depends on the slowly varying coordinate s. As s increases or when one follows the wave motion in the downstream direction the saddle point traces out a path in the complex η-plane as shown in figure 6. Now deform the η-integration contour in the complex η-plane to pass through the

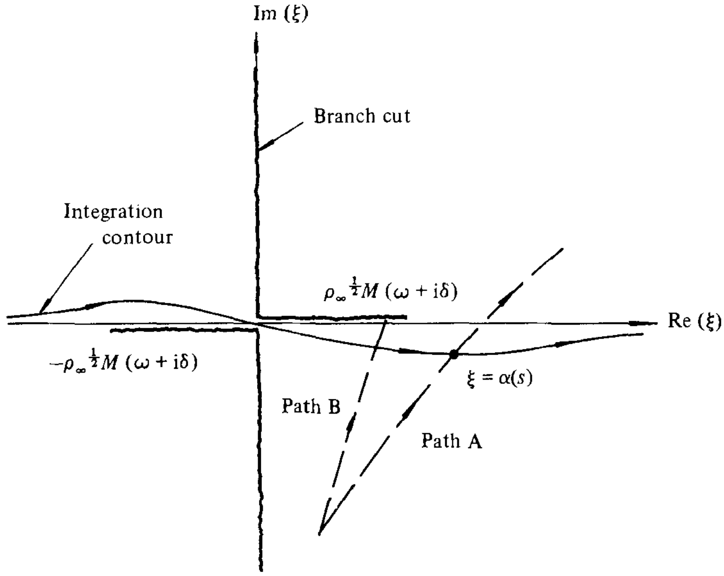

FIGURE 7. ξ-plane showing the branch cuts of $\lambda_-(\xi)$, the saddle point $\xi = \alpha(s)$ and its possible trajectories as s increases, and the deformed integration contour.

saddle point. The asymptotic expansion of the integral of (5.9) can easily be evaluated following the procedure of Dingle (1973). After some lengthy but straightforward algebra we find

$$p_0^{\mathrm{u}}(s,\tilde{y}) \underset{\substack{s,\,\tilde{y}\,\text{fixed}\\ \epsilon\to 0}}{\sim} \exp\left[\mathrm{i}\frac{\theta(s)}{\epsilon}+\mathrm{i}\lambda_+(\alpha)\,\tilde{y}\epsilon^{-1/N}\right]\Big\{\bar{A}_0(s)+\epsilon^{1-2/N}\frac{\mathrm{i}}{2}(\lambda_+')^2\alpha'\bar{A}_0\tilde{y}^2$$
$$+\epsilon^{1-1/N}(\tfrac{1}{2}\lambda_+''\alpha'\bar{A}_0+\lambda_+'\bar{A}_0')\,\tilde{y}+\epsilon\bar{A}_1(s)+O(\epsilon^{2-4/N})\Big\}, \quad (5.13)$$

where

$$\lambda_+' = \frac{\mathrm{d}\lambda_+}{\mathrm{d}\alpha}, \quad \lambda_+'' = \frac{\mathrm{d}^2\lambda_+}{\mathrm{d}\alpha^2}, \quad \alpha' = \frac{\mathrm{d}\alpha}{\mathrm{d}s}, \quad \bar{A}_0' = \frac{\mathrm{d}\bar{A}_0(s)}{\mathrm{d}s}.$$

Similarly in the region below the mixing layer, a lower intermediate solution can be found. On evaluating the integrals involved by means of the saddle-point method and assuming that the arbitrary function $\hat{A}(s,\epsilon)$ has an expansion of the form

$$\hat{A}(s,\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \hat{A}_n(s), \quad (5.14)$$

the following asymptotic expansion of the lower intermediate solution is derived:

$$p_0^{l}(s,\tilde{y}) \underset{\substack{s,\,\tilde{y}\,\text{fixed}\\ \epsilon\to 0}}{\sim} \exp\left[\mathrm{i}\frac{\theta(s)}{\epsilon}+\lambda_-(\alpha)\,\tilde{y}\epsilon^{-1/N}\right]\Big\{\hat{A}_0(s)-\epsilon^{1-2/N}\frac{\mathrm{i}}{2}(\lambda_-')^2\alpha'\hat{A}_0\tilde{y}^2$$
$$-\epsilon^{1-1/N}\mathrm{i}(\tfrac{1}{2}\lambda_-''\alpha'\hat{A}_0+\lambda_-'\hat{A}_0'-\omega\bar{v}_\infty\rho_\infty M^2\hat{A}_0)\,\tilde{y}+\epsilon\hat{A}_1(s)+O(\epsilon^{2-4/N})\Big\}, \quad (5.15)$$

where

$$\lambda_-' = \frac{\mathrm{d}\lambda_-(\alpha)}{\mathrm{d}\alpha}, \quad \lambda_-'' = \frac{\mathrm{d}^2\lambda_-(\alpha)}{\mathrm{d}\alpha^2}, \quad \hat{A}_0' = \frac{\mathrm{d}\hat{A}_0(s)}{\mathrm{d}s}.$$

Of interest later on is the path of the saddle point $\xi = \alpha(s)$ in the complex ξ-plane ($\xi = \epsilon k$). In general, two classes of trajectories are possible. They are shown as paths 'A' and 'B' in figure 7. The class typified by path 'A' has a trajectory which crosses the real ξ-axis to the right of the branch point $\rho_\infty^{\frac{1}{2}} M\omega$. The class typified by that of path 'B' follows a trajectory which reaches the real ξ-axis at the branch cut. We will temporarily assume that the saddle point follows path 'A' as s increases (the 'damped subsonic wave' case). In the case of path 'B' one encounters the 'damped supersonic wave' problem as discussed in §1. This case will be considered in §7.

Now we will take the intermediate limit of the one-term inner solution given by (3.13). On accounting for the behaviour of the function ζ_1 and ζ_2 as their argument $\to \pm\infty$ (see (3.8) and (3.12)) it is easy to find

$$\lim_{\epsilon\to 0} p_i \underset{s,\,\tilde{y}\,\text{fixed},\,\tilde{y}>0}{\sim} e^{i\theta(s)/\epsilon}[A_0(s)\,e^{i\lambda+\tilde{y}\epsilon^{1/N}}+B_0(s)\,e^{-i\lambda+\tilde{y}\epsilon^{1/N}}+O(\epsilon^{1-2/N})], \tag{5.16}$$

$$\lim_{\epsilon\to 0} p_i \underset{s,\,\tilde{y}\,\text{fixed},\,\tilde{y}<0}{\sim} e^{i\theta(s)/\epsilon}[(A_0(s)\,\beta_1(\alpha)+B_0(s)\,\beta_2(\alpha))\,e^{\lambda-\tilde{y}\epsilon^{1/N}} + (A_0(s)\,\bar{\beta}_1(\alpha)+B_0(s)\,\bar{\beta}_2(\alpha))\,e^{-\lambda-\tilde{y}\epsilon^{1/N}}+O(\epsilon^{1-2/N})]. \tag{5.17}$$

On comparing (5.16) with (5.13), matching to terms of order unity is achieved if

$$B_0(s) = 0, \tag{5.18}$$

and

$$\tilde{A}_0(s) = A_0(s). \tag{5.19}$$

Similarly on comparing (5.17) with (5.15), matching to terms of the lowest order is possible if

$$\bar{\beta}_1(\alpha) = 0, \tag{5.20}$$

and

$$\hat{A}_0(s) = A_0(s)\,\beta_1(\alpha). \tag{5.21}$$

Equation (5.18) requires that the one-term inner solution (3.13) should be a function of ζ_1 alone. Equations (5.19) and (5.21) relate the lowest-order amplitude function of the outer solutions to that of the inner solution. Equation (5.20) cannot be satisfied except for special values of α. These special values are the eigenvalues of the commonly used locally parallel-flow hydrodynamic stability theory. When the local eigenvalue $\alpha(s)$ is such that on following the instability wave motion downstream it traces out a trajectory of type 'A' as shown in figure 7, the one-term inner solution as given by (3.13) decays to zero as $y\to\pm\infty$. This is so despite the fact that the solution is actually not valid in the limit of unbounded values of y. In this case the inner solution is identical with the eigenfunction of the classical locally parallel-flow hydrodynamic stability theory. At this stage, except for the unknown amplitude $A_0(s)$, the one-term inner solution as well as the one-term outer solution have been completely determined. An equation for $A_0(s)$ will be found at the next stage of the matching process.

6. The wave-amplitude equation

The second term of the inner solution (to order ϵ) is given by the solution of (3.4) with $n = 1$:

$$\frac{\partial^2 p_1}{\partial y^2}+\left[\frac{2\alpha}{\omega-\alpha\bar{u}}\frac{\partial\bar{u}}{\partial y}-\frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial y}\right]\frac{\partial p_1}{\partial y}+[\bar{\rho}M^2(\omega-\alpha\bar{u})^2-\alpha^2]\,p_1 = \chi_1(s,y). \tag{6.1}$$

The inhomogeneous term χ_1 depends on the zeroth-order solution p_0 only. Except for the zeroth-order amplitude function $A_0(s)$, this solution has been found, so that we may regard χ_1 as a known function of s, y and A_0. The solution of the inhomogeneous equation (6.1) can be constructed by the method of variation of parameters (see Boyce & DiPrima 1977). In terms of the two linearly independent solutions of the homogeneous equation $\zeta_1(s,y)$ and $\zeta_2(s,y)$, which have been used in (3.9), this solution may be written as

$$p_1 = -\zeta_1(s,y)\int_{\hat{y}(s)}^{y}\frac{\chi_1(s,t)\,\zeta_2(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}+\zeta_2(s,y)\int_{\hat{y}(s)}^{y}\frac{\chi_1(s,t)\,\zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}+A_1(s)\,\zeta_1(s,y), \quad (6.2)$$

where $\hat{y}(s)$ is an arbitrary function of s. Without loss of generality, we will assume that $-y_{\mathrm{n}} < \hat{y} < y_{\mathrm{m}}$. $W(\zeta_1,\zeta_2)$ is the Wronskian and $A_1(s)$ is the ϵ-order amplitude function of the instability wave.

In the regions $y > y_{\mathrm{m}}$ and $y < -y_{\mathrm{n}}$ the two linearly independent solutions ζ_1 and ζ_2 are exponential functions of y given by (3.8) and (3.12). Thus in these regions, (6.2) may be written out explicitly in terms of simple functions of y. These expressions will be useful to the operation of matching the inner solution with the intermediate solution to be carried out later. After some algebraic manipulations it is straightforward to find

$$p_1 = [\tfrac{1}{2}\mathrm{i}(\lambda_+')^2\,\alpha' A_0\,y^2+(\tfrac{1}{2}\lambda_+''\,\alpha' A_0+\lambda_+' A_0')\,y+D]\,\mathrm{e}^{\mathrm{i}\lambda_+ y}+E\,\mathrm{e}^{-\mathrm{i}\lambda_+ y} \quad (y > y_{\mathrm{m}}), \quad (6.3a)$$

where

$$D = A_1+\int_{\hat{y}}^{y_{\mathrm{m}}}\frac{\chi_1(s,t)\,\zeta_2(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}+\int_{y_{\mathrm{m}}}\frac{\chi_1(s,t)\,\zeta_2(s,t)}{W(\zeta_1,\zeta_2)}\mathrm{d}t-\frac{1}{2\mathrm{i}\lambda_+}(\tfrac{1}{2}\lambda_+''\,\alpha' A_0+\lambda_+' A_0'), \quad (6.3b)$$

$$E = \int_{\hat{y}}^{y_{\mathrm{m}}}\frac{\chi_1(s,t)\,\zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}+\int_{y_{\mathrm{m}}}\frac{\chi_1(s,t)\,\zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}; \quad (6.3c)$$

$$p_1 = [-\tfrac{1}{2}\mathrm{i}(\lambda_-')^2\,\alpha'\hat{A}_0\,y^2-\mathrm{i}(\tfrac{1}{2}\lambda_-''\,\alpha'\hat{A}_0+\lambda_-'\hat{A}_0'-\omega\bar{v}_\infty\,\rho_\infty\,M^2\hat{A}_0)\,y+F]\,\mathrm{e}^{\lambda_- y}+G\,\mathrm{e}^{-\lambda_- y} \quad (y < -y_{\mathrm{n}}), \quad (6.4a)$$

where

$$F = \beta_1 A_1-\beta_1\int_{\hat{y}}^{-y_{\mathrm{n}}}\frac{\chi_1(s,t)\,\zeta_2(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}-\frac{1}{2\lambda_1}\int^{-y_{\mathrm{n}}}\chi_1(s,t)\,\mathrm{e}^{-\lambda_1 t}\,\mathrm{d}t+\beta_2\int_{\hat{y}}^{-y_{\mathrm{n}}}\frac{\chi_1(s,t)\,\zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}-\frac{1}{4\lambda_-^2}(-2\mathrm{i}\alpha\hat{A}_0'-\mathrm{i}\alpha'\hat{A}_0-\mathrm{i}(\lambda_-')^2\alpha'\hat{A}_0-2\mathrm{i}\omega\bar{v}_\infty\,\rho_\infty\,M^2\hat{A}_0), \quad (6.4b)$$

$$G = \bar{\beta}_2\left[\int_{\hat{y}}^{-y_{\mathrm{n}}}\frac{\chi_1(s,t)\,\zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}-\int^{-y_{\mathrm{n}}}\frac{\chi_1(s,t)\,\zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}\right]. \quad (6.4c)$$

On substituting (3.9), (6.2), (5.18), (5.19) and (5.21) into (3.1), the two-term inner solution is found. This solution must match the intermediate asymptotic expansions (5.13) and (5.15) in the intermediate limit. By means of (6.3) and (6.4) the intermediate limit of this two-term inner solution can readily be calculated:

$$\lim_{\substack{\epsilon\to 0\\ s,\,\tilde{y}\,\text{fixed}\\ (\tilde{y}>0)}} p_{\mathrm{i}} \sim \mathrm{e}^{\mathrm{i}\theta(s)/\epsilon}\{\mathrm{e}^{\mathrm{i}\lambda_+\tilde{y}\epsilon^{1/N}}[\tilde{A}_0+\epsilon^{1-2/N}\tfrac{1}{2}\mathrm{i}(\lambda_+')^2\,\alpha'\tilde{A}_0\tilde{y}^2+\epsilon^{1-1/N}(\tfrac{1}{2}\lambda_+''\,\alpha'\tilde{A}_0+\lambda_+'\tilde{A}_0')\,\tilde{y}+\epsilon D]+\epsilon E\,\mathrm{e}^{-\mathrm{i}\lambda_+\tilde{y}\epsilon^{1/N}}+O(\epsilon^{2-4/N})\}, \quad (6.5)$$

$$\lim_{\substack{\epsilon\to 0\\ s,\,\tilde{y}\,\text{fixed}\\ (\tilde{y}<0)}} p_{\mathrm{i}} \sim \mathrm{e}^{\mathrm{i}\theta(s)/\epsilon}\{\mathrm{e}^{\lambda_-\tilde{y}\epsilon^{1/N}}[\hat{A}_0-\epsilon^{1-2/N}\tfrac{1}{2}\mathrm{i}(\lambda_-')^2\,\alpha'\hat{A}_0\tilde{y}^2-\epsilon^{1-1/N}\mathrm{i}(\tfrac{1}{2}\lambda_-''\,\alpha'\hat{A}_0+\lambda_-'\hat{A}_0'-\omega\bar{v}_\infty\,\rho_\infty\,M^2\hat{A}_0)\,\tilde{y}+\epsilon F]+\epsilon G\,\mathrm{e}^{-\lambda_-\tilde{y}\epsilon^{1/N}}+O(\epsilon^{2-4/N})\}, \quad (6.6)$$

where D, E, F and G are the same as in (6.3) and (6.4).

To terms up to order $\epsilon^{1-1/N}$, (6.5) agrees with the upper intermediate asymptotic expansion (5.15). Similarly to the same order, (6.6) and the lower intermediate asymptotic expansion (5.16) are equal. Now to match terms to order ϵ we must have

$$\tilde{A}_1 = D, \tag{6.7}$$

$$E = 0, \tag{6.8}$$

$$\hat{A}_1 = F, \tag{6.9}$$

$$G = 0, \tag{6.10}$$

Equations (6.7) and (6.9) define the values of the unknown function $\tilde{A}_1$ and $\hat{A}_1$ in terms of known functions of s and $A_1(s)$. Equations (6.8) and (6.10) provide two equations for the two unknowns $A_0(s)$ and $\hat{y}(s)$. (Note that χ_1 is a linear function of A_0 and $A_0'(s)$.) By subtracting one of these equations from the other to form $E-G/\bar{\beta}_2 = 0$, a single equation for $A_0(s)$ is obtained:

$$\int_{-y_n}^{y_m} \frac{\chi_1(s,t), \zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)} + \int_{y_m} \frac{\chi_1(s,t)\,\zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)} + \int^{-y_n} \frac{\chi_1(s,t)\,\zeta_1(s,t)}{W(\zeta_1,\zeta_2)}\mathrm{d}t = 0. \tag{6.11}$$

On writing out the above equation in full, one obtains a first-order ordinary differential equation for A_0 in the form

$$I_0\frac{\mathrm{d}A_0}{\mathrm{d}s} + I_1 A_0 = 0. \tag{6.12}$$

In (6.12) I_0 and I_1 are known functions of s. They are, however, quite complicated and will not be given here. The solution of (6.2) is

$$A_0(s) = \bar{A}\exp\left[-\int_{s_0}^{s}\frac{I_1}{I_0}\mathrm{d}s\right]. \tag{6.13}$$

With the instability-wave amplitude $A_0(s)$ determined, the complete solution of the present problem (both inner and outer solutions) to order unity is now found. The ϵ-order wave amplitude, namely $A_1(s)$, is still not specified at this stage. It will be decided by matching terms to order ϵ^2.

It is interesting and important to point out at this time that in the case of 'damped subsonic waves' (6.11) is identical with the solvability condition of the multiple-scales solution of Tam & Morris (1980). To show this, it can be verified under the assumption of a 'damped subsonic wave' that the second and third terms of (6.11) are equal to the following integrals:

$$\int_{y_m} \frac{\chi_1(s,t)\,\zeta_1(s,t)}{W(\zeta_1,\zeta_2)}\mathrm{d}t = \int_{y_m}^{\infty} \frac{\chi_1(s,t)\,\zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)}, \tag{6.14}$$

$$\int^{-y_n} \frac{\chi_1(s,t)\,\zeta_1(s,t)\,\mathrm{d}t}{W(\zeta_1,\zeta_2)} = \int_{-\infty}^{-y_n} \frac{\chi_1(s,t)\,\zeta_1(s,t)}{W(\zeta_1,\zeta_2)}\mathrm{d}t. \tag{6.15}$$

By combining these two equations with (6.11) we find

$$\int_{-\infty}^{\infty} \frac{\chi_1(s,t)\,\zeta_1(s,t)}{W(\zeta_1,\zeta_2)}\mathrm{d}t = 0. \tag{6.16}$$

The Wronskian $W(\zeta_1,\zeta_2)$ of (6.1) is equal to a constant times $\bar{\rho}(\omega-\alpha\bar{u})^2$. In this form it is easy to see that (6.16) is identical with the solvability condition given by equation (2.19) of Tam & Morris (1980). Therefore in the case of a 'damped subsonic wave' the solution obtained by the method of matched asymptotic expansions and that obtained by the method of multiple scales are the same, at least to order ϵ. We expect the two methods would give identical results for the higher-order terms as well.

7. Resolution of the supersonic damped wave problem

When the flow is highly supersonic the phenomenon of the 'damped supersonic wave' described in §§1 and 5 will inevitably occur. The basic problem here is that as soon as the locally parallel instability-wave solution having supersonic phase velocity relative to the ambient speed of sound becomes damped it is not possible to find a solution of the same family that remains bounded in the limit $|y| \to \infty$. Within the framework of the classical hydrodynamic stability theory and the more recent theories based on the method of multiple scales (e.g. Saric & Nayfeh 1975; Crighton & Gaster 1976; Plaschko 1979; Tam & Morris 1980; Morris 1981) this boundedness condition is an essential requirement in the formulation of the lowest order or the instability-wave solution. Failure to meet this specification renders these methods rather useless. The reason which leads to the breakdown of these approaches is not too difficult to find. Earlier Tam & Morris (1980) have shown that the multiple-scales instability-wave solution is not uniformly valid for large $|y|$. It should therefore not be too surprising that the solution exhibits undesirable behaviour in this region. From the standpoint of the method of matched asymptotic expansions, the imposition of the boundedness condition on the instability-wave solution at $|y| \to \infty$ appears to be somewhat unreasonable and unnecessary. Here the locally parallel-flow instability wave solution is just the first-term inner solution. This solution, however, does not represent the physical wave motion at large $|y|$. In this region only the outer solution would provide the correct description of the wave field. Requiring a solution to satisfy an imposed boundary condition in a region where it does not represent the physical entity naturally would lead to unsurmountable difficulties. In this sense the failure of the method of multiple scales to produce a valid global solution is not entirely unexpected.

In constructing the spatially growing instability-wave solution by the method of matched asymptotic expansions the imposition of the boundedness or outgoing-wave condition at $|y| \to \infty$ is never a problem. This condition can always be satisfied by the outer solutions. Within this method, the 'damped supersonic wave' phenomenon therefore no longer manifests itself in the boundedness condition at $|y| \to \infty$, but instead shows up in the process of matching solutions. As discussed in §5, when it occurs the trajectory of the saddle point $\xi = \alpha(s)$ in the complex ξ-plane runs into the branch cut of the function $\lambda_-(\xi)$ as illustrated by path 'B' of figure 7. This problem can, however, be easily resolved by the use of analytic continuation. Suppose the saddle point follows path 'B' of figure 7 as s increases. Upon reaching the branch cut of λ_- at the positive real ξ-axis, the trajectory, according to the theory of complex variables, may be analytically continued into the second Riemann sheet as shown in figure 8. This is permissible because boundedness of solution at large $|y|$ is no longer a requirement of the inner solution. To evaluate the ξ-integral by the saddle-point method it will now be necessary to deform the integration contour analytically into the second Riemann sheet to pass through the saddle point as well. This can be done, however, without changing the form of the asymptotic expansions at all. With the above modification of the saddle-point trajectory it is easy to see that the inner and outer (or the intermediate) solutions obtained in the previous

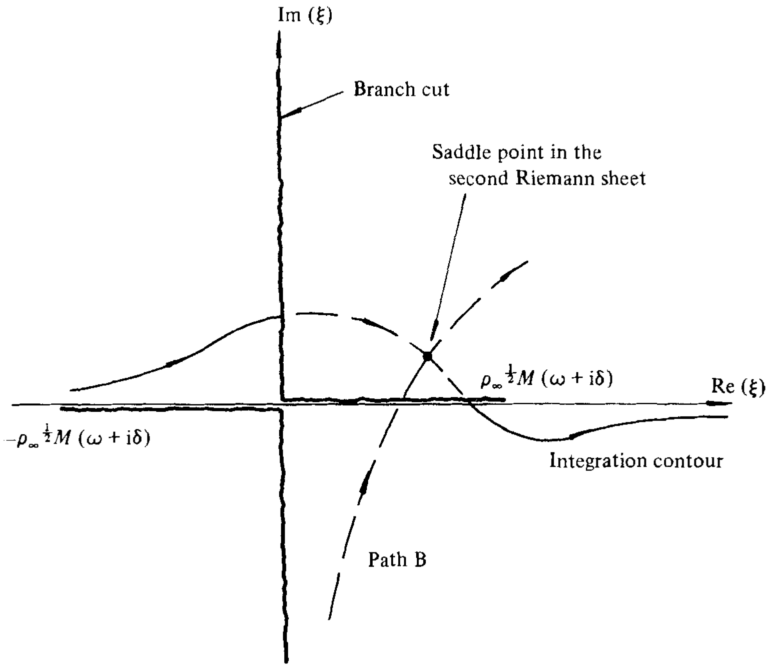

FIGURE 8. Complex ξ-plane showing analytic continuation of the saddle point $\xi = \alpha(s)$ and the integration contour into the second Riemann sheet for supersonic damped waves.

sections will remain valid even for very high supersonic-Mach-number flows. In other words, the 'damped supersonic wave' phenomenon poses no real obstacle to the construction of a global instability–acoustic-wave solution by the method of matched asymptotic expansions.

8. Concluding remarks

At the end of §6 it was pointed out that for 'damped subsonic waves' the method of multiple scales and the method of matched asymptotic expansions when applied to the problem of spatially growing instability waves gave identical results. Despite this fact, we wish to emphasize that these two methods are actually based on two fundamentally opposite points of view. The method of multiple scales is rooted primarily in the supposition that the instability wave is a local event in the mixing layer. It is only when the growth and decay of the wave amplitude in the flow direction are considered in their totality that the global picture of the entire wave phenomenon emerges. Thus the method is predicated on the existence of a local instability-wave eigenvalue solution. The wave-amplitude equation which describes the global nature of the wave is obtained from the solvability condition of the basic eigenvalue problem. Physically this solvability condition may be interpreted as the requirement that the local nature of the instability-wave solution is not to be violated. In this way the global character of the solution is treated by this method only as a secondary consequence of the local instability wave phenomenon. On the other hand the method of matched asymptotic expansions is based on the premise that the instability wave and its associated noise field form a global problem. Therefore, even at the very beginning, the problem must be considered in its entirety in the whole physical space. Although, of necessity, the full solution is divided into an inner solution which eventually gives the instability wave and an outer solution which gives the far-field radiated sound, they are to be considered to be of equal importance. In fact the two solutions are mutually dependent on each other. In this method the local nature of the instability wave is revealed only in the matching process. A closer examination of the steps taken in matching the intermediate and the inner expansions as described in §§5 and 6 shows that only a small portion of the total information contained in the outer solution is used. Primarily it is the parts of the integrand near the saddle points of the outer solutions (4.3)–(4.6) (see figures 6 and 7) which are involved in the matching. The radiated sound field, which can be shown to be given by the contributions of the integrand near the branch cuts, does not play any vital role. In spite of this, matching is a global process requiring the agreement of the inner and the intermediate (or outer) solutions over an extended region of space. Matching terms to order unity requires the existence of an instability-wave solution over the whole length of the mixing layer in the flow direction. Matching terms to order ϵ provides the global amplitude distribution of the wave and its associated acoustic field. Now if the flow is subsonic or incompressible the disturbances associated with the entire phenomenon are confined to the immediate neighbourhood of the mixing layer. In this case both methods should be applicable, as in a sense the problem is still local in nature. However, when the flow is highly supersonic the radiated sound field becomes increasingly important. It now extends all the way to infinity. When this is the case, the problem is definitely global in nature. Based on the above discussion, clearly in this instance the only logical way to analyse the problem is by the method of matched asymptotic expansions of this paper.

Finally we would like to mention again that the acoustic field associated with the spatial evolution of the instability wave is embedded in the outer solution. If one is interested in the radiated acoustic field it is only necessary to perform an asymptotic evaluation of the k-integral of the outer solution under the condition $r = (x^2+y^2)^{\frac{1}{2}} \rightarrow \infty$. This can be done by the well-known method of stationary phase. We will not carry out this analysis here as it is not the main point of interest of the plane mixing-layer problem. A detailed calculation of the radiated sound field for an axisymmetric supersonic jet will, however, be given in Part 2 (Tam & Burton 1984).

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